Noname manuscript No.
(will be inserted by the editor)

Temporal dynamics and nonclassical photon statistics of quadratically coupled optomechanical systems

Shailendra Kumar Singh · S. V. Muniandy

the date of receipt and acceptance should be inserted later

Abstract Quantum Optomechanical system serves as an interface for coupling between photons and phonons due to mechanical oscillations. We use the Heisenberg-Langevin approach under Markovian white noise approximation to study a quadratically coupled optomechanical system which contains a thin dielectric membrane quadratically coupled to the cavity field. A decorrelation method is employed to solve for a larger number of coupled equations. Transient mean number of cavity photons and phonons that provide dynamical behaviour are computed under different coupling regime. We have also obtained the two-boson second-order correlation functions for the cavity field, membrane oscillator and their cross correlations that provide nonclassical properties governed by quadratic optomechanical system.

1 INTRODUCTION

Recent years have seen significant experimental progress in realizing deterministic interactions between single photons, which has profound importance for future optical technologies. Most novel experiments have been explored in cavity quantum electrodynamics (cQED) [1,2], where photons inherent the saturation of a single two-level atom due to strong interactions between the atom and the cavity field. A single atom-cavity system described by well-known Jaynes-Cummings model [3] has been used as an important test bed for implementation of quantum information algorithms as well as construction of a quantum network with the aim for quantum computation. The strong-coupling regime of the cQED is reached when the coupling strength of the atom with the cavity mode dominates over the decoherence processes, due to spontaneous emission from the excited atomic level

Corresponding Author: Shailendra Kumar Singh
Institute of Nuclear Sciences, Hacettepe University, 06800, Ankara, Turkey
E-mail: singhshailendra3@gmail.com

S.V. Muniandy
Center of Theoretical and Computational Physics, Department of Physics, University of Malaya, 50603 KualaLumpur, Malaysia
E-mail: msithi.um.edu.my

to ground state and the leakage of photons in the cavity mode. Based on Fabry-Perot interferometry [4], semiconductor microcavities have been developed where excitons act as quantum systems [5]. Alternative approaches have been explored based on slow-light enhanced Kerr nonlinearites [6], single dye-molecules [7], strong photon interactions mediated by Rydberg atoms [8], atoms coupled to hollow-core fibers [9] and nanofiber traps [10].

Optomechanical system provides a more promising approach towards realizing strong photon interactions [11]. In comparison to atomic systems, for example macroscopic mechanical resonators are relatively easy to control and to be integrated with other systems. In these systems, an optical cavity, with a movable end mirror [12, 13] or with a micromechanical membrane is subjected to mechanical effect caused by light through radiation pressure [14]. So, Cavity Quantum Optomechanics has emerged as an interesting area for studying quantum features at the mesoscale, where it is possible to control the quantum state of mechanical oscillators by their coupling to light field [15]. Recent advances in this area includes the realization of quantum-coherent coupling of a mechanical oscillator with an optical cavity [16], where the coupling rate exceeds both the cavity and mechanical motion decoherence rate, laser cooling of a nanomechanical oscillator to its ground state [17].

Experimental and theoretical proposals aiming to study quantum aspects of interactions between the optical cavities and mechanical objects have focused on cavities in which one of the cavity's mirror is free to move for example, in response to radiation pressure exerted by light in the cavity. This nearly includes all optomechanical systems described in the literature, including cavities with 'folded' geometries, cavities in which multiple mirrors are free to move [18], and whispering gallery mode resonators [19] in which light is confined to a waveguide. All these methods have two important considerations. First, cavity's detuning is proportional to the displacement of a mechanical degree of freedom, which is mirror displacement or waveguide elongation. Second, a single device must provide both optical confinement and mechanical effects. So, achieving good mechanical and optical properties both simultaneously has been the main aim in quantum optomechanical systems because high-finesse mirrors are not easily integrated into micromachined devices. Experimental work done so far have achieved sufficient optomechanical coupling to laser-cooled mechanical devices [20, 21] but have not yet reached to quantum regime due to technical problems mentioned above. A more fundamental challenge is to read out the mechanical element's energy eigenstate. Displacement measurements cannot determine an oscillator's energy eigenstate and measurements coupling to quantities other than displacement have been difficult to realize in practice [22, 23].

In this work, we theoretically study an optomechanical system realized in seminal experimental works [24, 25] and has potential to resolve both of these abovementioned challenges. We study a system which contains a thin dielectric membrane with quadratic response to the cavity fields. Coupled Heisenberg-Langevin equations are obtained under Markovian white noise approximation, and solutions up to second-order correlation operators have been developed in Section II. In the same section, we also discuss the transient dynamics of the system Hamiltonian

in various scenario. Section III discusses the results with the nonclassical photon statistics of the cavity mode as well as mechanical oscillations mode including cross-correlation between them. We conclude our results in Section IV.

2 Heisenberg-Langevin Approach For Optomechanics

We consider a quadratically coupled optomechanical resonator formed by a micropillar with moveable Bragg reflectors and a thin dielectric membrane at the node (or antinode) of the resonator. The mechanical displacement of the membrane quadratically couples to the cavity photon number. In addition, we have a monochromatic laser field with frequency ω_L applied to drive the cavity. The corresponding Hamiltonian for the scheme is given by,

$$\hat{H} = \hat{H}_0 + \hat{H}_R, \tag{1}$$

where $\hat{H}_0$ and $\hat{H}_R$ are given by Eqs. (2) and (3). In our model, $\hat{H}_R$ is the part where all the two subsystem cavity photons, phonons (due to mechanical motion of the membrane) respectively interact with their corresponding reservoirs which are collection of harmonic oscillators. We write

$$\hat{H}_0 = \omega_c \hat{a}^\dagger \hat{a} + \omega_M \hat{b}^\dagger \hat{b} + g_{opt}\left[\hat{a}^\dagger \hat{a}\left(\hat{b}^\dagger + \hat{b}\right)^2\right] + \Omega\left[e^{-i\omega_L t}\hat{a}^\dagger \ + e^{i\omega_L t}\hat{a}\right] \tag{2}$$

where $\hat{a}$ and $\hat{b}$ correspond to field operators for cavity photons and phonons (due to mechanical motion of the membrane) with frequencies ω_c and ω_M, respectively. The third term in R.H.S of Eq. (2) describes the quadratic optomechanical coupling with strength g_{opt} between the cavity field and the mechanical motion of the membrane and the fourth term describes driving process of the cavity field, with Ω is the Rabi frequency of the driving field. Meanwhile, $\hat{H}_R$ takes the following form:

$$\hat{H}_R = \overbrace{\sum_k \omega_k \hat{m}_k^\dagger \hat{m}_k + \sum_k g_k\left(\hat{m}_k^\dagger \hat{a} + \hat{a}^\dagger \hat{m}_k\right)}^{\text{for cavity mode}} + \overbrace{\sum_{k'} \omega_{k'} \hat{n}_{k'}^\dagger \hat{n}_{k'} + \sum_{k'} g_{k'}\left(\hat{n}_{k'}^\dagger \hat{b} + \hat{b}^\dagger \hat{n}_{k'}\right)}^{\text{for moving membrane}}, \tag{3}$$

where the first two terms on the R.H.S of Eq.(3) represent the damping of cavity mode through reservoir of harmonic oscillator operators $\hat{m}_k$ and $\hat{m}_k^\dagger$; while the third and fourth terms are responsible for damping of membrane through the harmonic oscillator operators $\hat{n}_{k'}$ and $\left(\hat{n}_{k'}^\dagger\right)$. In the rotating frame with the driving frequency ω_L, the Hamiltonian $\hat{H}_0$ reduces to

$$\hat{H}_0 = \Delta_c \hat{a}^\dagger \hat{a} + \omega_M \hat{b}^\dagger \hat{b} + g_{opt}\left[\hat{a}^\dagger \hat{a}\left(\hat{b}^\dagger + \hat{b}\right)^2\right] + \Omega\left(\hat{a}^\dagger \ + \hat{a}\right), \tag{4}$$

where $\Delta_c = (\omega_c - \omega_L)$ is the corresponding detuning from the driving laser frequency ω_L. Expanding the quadratic term $g_{opt}\left[\hat{a}^\dagger \hat{a}\left(\hat{b}^\dagger + \hat{b}\right)^2\right]$, we have the Hamiltonian $\hat{H}_0$ in the following form:

$$\hat{H}_0 = \Delta_c \hat{a}^\dagger \hat{a} + \omega_M \hat{b}^\dagger \hat{b} + g_{opt}\left[\hat{a}^\dagger \hat{a}\left(\hat{b}^{\dagger 2} + \hat{b}^2 + 2\hat{b}^\dagger \hat{b} + 1\right)\right] + \Omega\left(\hat{a}^\dagger \ + \hat{a}\right) \tag{5}$$

Coupling strength g_{opt} should satisfy the condition $(\omega_M + 4sg_{opt}) > 0$ for the stability of the membrane [26], where s is the number of photons inside the cavity. The corresponding Heisenberg equations of motion for the field operators of different subsystem (cavity, membrane) are given by Eqs. (6) and (8), whereas the corresponding equations of motion for reservoir operators are given by Eqs. (7) and (9):

$$\frac{d}{dt}\hat{a} = -i\Delta_c\hat{a} - ig_{opt}\left[\hat{a}\left(\hat{b}^{\dagger 2} + \hat{b}^2 + 2\hat{b}^\dagger\hat{b} + 1\right)\right] - i\Omega - i\sum_k g_k\hat{m}_k. \tag{6}$$

$$\frac{d}{dt}\hat{m}_k = -i\omega_k\hat{m}_k - ig_k\hat{a}. \tag{7}$$

$$\frac{d}{dt}\hat{b} = -i\omega_M\hat{b} - 2ig_{opt}\left[\hat{a}^\dagger\hat{a}\left(\hat{b} + \hat{b}^\dagger\right)\right] - i\sum_{k'} g_{k'}\hat{n}_{k'}. \tag{8}$$

$$\frac{d}{dt}\hat{n}_{k'} = -i\omega_{k'}\hat{n}_{k'} - ig_{k'}\hat{b}. \tag{9}$$

Now the corresponding operator equations for different reservoir need to be explicitly solved further to remove the terms of reservoir operators in Eqs (6) and (8). Integrating Eq. (7), one gets

$$\hat{m}_k(t) = \hat{m}_k(0)e^{-i\omega_k t} - ig_k\int_0^t dt^{'}\,\hat{a}(t^{'})e^{-i\omega_k(t-t^{'})}. \tag{10}$$

In Eq. (10), the first term represents the free evolution of the reservoir modes, whereas the second term arises from their interaction with the cavity. The reservoir modes $\hat{m}_k(t)$ can be eliminated from the Eq. (6) by substituting the value of $\hat{m}_k(t)$ from Eq. (10) in Eq. (6). So, we have

$$\frac{d}{dt}\hat{a} = -i\Delta_c\hat{a} - ig_{opt}\left[\hat{a}\left(\hat{b}^{\dagger 2} + \hat{b}^2 + 2\hat{b}^\dagger\hat{b} + 1\right)\right] - i\Omega - \sum_k g_k^2\int_0^t dt^{'}\,\hat{a}(t^{'})e^{-i\omega_k(t-t^{'})} + F_a(t), \tag{11}$$

where $F_a(t) = -i\sum_k g_k\hat{m}_k(0)e^{-i\omega_k t}$ is the noise operator for the cavity which depends on the reservoir variables. The term containing the integral is expressed as

$$\sum_k g_k^2\int_0^t dt^{'}\,\hat{a}(t^{'})e^{-i\omega_k(t-t^{'})} \simeq \frac{1}{2}\Gamma_a\hat{a}(t). \tag{12}$$

Thus, from Eqs. (11) and (12), we obtain

$$\frac{d}{dt}\hat{a} = -i\Delta_c\hat{a} - ig_{opt}\left[\hat{a}\left(\hat{b}^{\dagger 2} + \hat{b}^2 + 2\hat{b}^\dagger\hat{b} + 1\right)\right] - i\Omega - \frac{1}{2}\Gamma_a\ \hat{a}(t) + F_a(t), \tag{13}$$

where Γ_a is the damping constant for cavity mode. Similarly, we have equation for mechanical motion of the membrane as follows:

$$\frac{d}{dt}\hat{b} = -i\omega_M\hat{b} - 2ig_{opt}\left[\hat{a}^\dagger\hat{a}\left(\hat{b} + \hat{b}^\dagger\right)\right] - \frac{1}{2}\Gamma_b\ \hat{b} + F_b(t); \tag{14}$$

where $F_b(t)$ and Γ_b are noise operator and damping constant for moving membrane respectively, given by

$$F_b(t) = -i\sum_{k'} g_{k'}\hat{n}_{k'}(0)e^{-i\omega_{k'}t}, \tag{15}$$

and

$$\sum_{k'} g_{k'}^2 \int_0^t dt^{'} \hat{b}(t^{'}) e^{-i\omega_{k'}(t-t^{'})} \simeq \frac{1}{2}\Gamma_b \hat{b}(t). \tag{16}$$

2.1 Quantum Correlations

We have obtained the coupled equations involving the mean of the operators $\hat{a}$ and $\hat{b}$, their corresponding adjoints and their odd and even products. Since both the reservoirs are big as well as always in thermal equilibrium, we have $\langle F_a(t)\rangle = \langle F_b(t)\rangle = 0$. We also have, $\left\langle F_{\hat{a}^\dagger}(t)\hat{b}(t)\right\rangle = \langle F_{\hat{b}}(t)\ \hat{a}(t)\rangle = \left\langle F_a(t)\hat{b}^\dagger(t)\right\rangle = \langle F_{\hat{b}^\dagger}(t)\ \hat{a}(t)\rangle = 0$. Similarly, quantum correlations between any subsystem (cavity, moving membrane) with different noise operator other than its own coupled reservoir's noise operator vanish altogether. The following set of equations has been obtained by using the Heisenberg-Langevin approach given in [27].

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= -i\Delta_c\langle\hat{a}\rangle - ig_{opt}\left[\left\langle\hat{a}\hat{b}^{\dagger 2}\right\rangle + \left\langle\hat{a}\hat{b}^2\right\rangle + 2\left\langle\hat{a}\hat{b}^\dagger\hat{b}\right\rangle + \langle\hat{a}\rangle\right]\\ &\quad -i\Omega - \frac{1}{2}\Gamma_a\langle\hat{a}\rangle.\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{a}^\dagger\right\rangle &= i\Delta_c\left\langle\hat{a}^\dagger\right\rangle + ig_{opt}\left[\left\langle\hat{a}^\dagger\hat{b}^{\dagger 2}\right\rangle + \left\langle\hat{a}^\dagger\hat{b}^2\right\rangle + 2\left\langle\hat{a}^\dagger\hat{b}^\dagger\hat{b}\right\rangle + \left\langle\hat{a}^\dagger\right\rangle\right]\\ &\quad +i\Omega - \frac{1}{2}\Gamma_a\left\langle\hat{a}^\dagger\right\rangle.\end{aligned} \tag{18}$$

$$\frac{d}{dt}\left\langle\hat{b}\right\rangle = -i\omega_M\left\langle\hat{b}\right\rangle - 2ig_{opt}\left[\left\langle\hat{a}^\dagger\hat{a}\hat{b}\right\rangle + \left\langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\right\rangle\right] - \frac{1}{2}\Gamma_b\left\langle\hat{b}\right\rangle. \tag{19}$$

$$\frac{d}{dt}\left\langle\hat{b}^\dagger\right\rangle = i\omega_M\left\langle\hat{b}^\dagger\right\rangle + 2ig_{opt}\left[\left\langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\right\rangle + \left\langle\hat{a}^\dagger\hat{a}\hat{b}\right\rangle\right] - \frac{1}{2}\Gamma_b\left\langle\hat{b}^\dagger\right\rangle. \tag{20}$$

$$\frac{d}{dt}\left\langle\hat{a}^\dagger\hat{a}\right\rangle = -i\Omega\left(\left\langle\hat{a}^\dagger\right\rangle - \langle\hat{a}\rangle\right) - \Gamma_a\left\langle\hat{a}^\dagger\hat{a}\right\rangle + \Gamma_a\bar{n}_{th}^a. \tag{21}$$

$$\frac{d}{dt}\left\langle\hat{b}^\dagger\hat{b}\right\rangle = -2ig_{opt}\left[\left\langle\hat{a}^\dagger\hat{a}\ \hat{b}^{\dagger 2}\right\rangle - \left\langle\hat{a}^\dagger\hat{a}\ \hat{b}^2\right\rangle\right] - \Gamma_b\left\langle\hat{b}^\dagger\hat{b}\right\rangle + \Gamma_b\bar{n}_{th}^b. \tag{22}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{a}\hat{b}^\dagger\right\rangle &= i(\omega_M - \Delta_c)\left\langle\hat{a}\hat{b}^\dagger\right\rangle - i\Omega\left\langle\hat{b}^\dagger\right\rangle - \frac{1}{2}(\Gamma_a + \Gamma_b)\left\langle\hat{a}\hat{b}^\dagger\right\rangle\\ &\quad +ig_{opt}\left[2\left\langle\hat{a}^\dagger\hat{a}^2\hat{b}\right\rangle - \left\langle\hat{a}\hat{b}^\dagger\hat{b}^2\right\rangle - \left\langle\hat{a}\hat{b}^{\dagger 3}\right\rangle + 2\left(\left\langle\hat{a}^\dagger\hat{a}^2\hat{b}^\dagger\right\rangle - \left\langle\hat{a}\hat{b}^{\dagger 2}\hat{b}\right\rangle\right) - \left\langle\hat{a}\hat{b}^\dagger\right\rangle\right].\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{a}^\dagger\hat{b}\right\rangle &= i(\Delta_c - \omega_M)\left\langle\hat{a}^\dagger\hat{b}\right\rangle + i\Omega\left\langle\hat{b}\right\rangle - \frac{1}{2}(\Gamma_a + \Gamma_b)\left\langle\hat{a}^\dagger\hat{b}\right\rangle\\ &\quad +ig_{opt}\left[\left\langle\hat{a}^\dagger\hat{b}^{\dagger 2}\hat{b}\right\rangle + \left\langle\hat{a}^\dagger\hat{b}^3\right\rangle - 2\left\langle\hat{a}^{\dagger 2}\hat{a}\hat{b}^\dagger\right\rangle + 2\left(\left\langle\hat{a}^\dagger\hat{b}^\dagger\hat{b}^2\right\rangle - \left\langle\hat{a}^{\dagger 2}\hat{a}\hat{b}\right\rangle\right) + \left\langle\hat{a}^\dagger\hat{b}\right\rangle\right].\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{a}\hat{b}\right\rangle &= -i(\Delta_c + \omega_M)\left\langle\hat{a}\hat{b}\right\rangle - i\Omega\left\langle\hat{b}\right\rangle - \frac{1}{2}(\Gamma_a + \Gamma_b)\left\langle\hat{a}\hat{b}\right\rangle - ig_{opt}\left[2\left\langle\hat{a}\hat{b}^\dagger\right\rangle + \left\langle\hat{a}\hat{b}^{\dagger 2}\hat{b}\right\rangle\right.\\ &\quad \left. +2\left\langle\hat{a}^\dagger\hat{a}^2\hat{b}^\dagger\right\rangle + \left\langle\hat{a}\hat{b}^3\right\rangle + 2\left(\left\langle\hat{a}\hat{b}\right\rangle + \left\langle\hat{a}\hat{b}^\dagger\hat{b}^2\right\rangle + \left\langle\hat{a}^\dagger\hat{a}^2\hat{b}\right\rangle\right) + \left\langle\hat{a}\hat{b}\right\rangle\right].\end{aligned} \tag{25}$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle = i\left(\Delta_c+\omega_M\right)\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle + i\Omega\left\langle \hat{b}^{\dagger}\right\rangle - \frac{1}{2}\left(\Gamma_a+\Gamma_b\right)\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle + ig_{opt}\left[2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{b}^{2}\right\rangle \right.$$
$$\left. +2\left\langle \hat{a}^{\dagger 2}\hat{a}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger 3}\right\rangle + 2\left(\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger 2}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger 2}\hat{a}\hat{b}^{\dagger}\right\rangle\right) + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\right]. \quad (26)$$

$$\frac{d}{dt}\left\langle \hat{a}^{2}\right\rangle = -2i\Delta_c\left\langle \hat{a}^{2}\right\rangle - 2i\Omega\left\langle \hat{a}\right\rangle - \Gamma_a\left\langle \hat{a}^{2}\right\rangle - 2ig_{opt}\left[\left\langle \hat{a}^{2}\hat{b}^{\dagger 2}\right\rangle + \left\langle \hat{a}^{2}\hat{b}^{2}\right\rangle + 2\left\langle \hat{a}^{2}\hat{b}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{a}^{2}\right\rangle\right]. \quad (27)$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger 2}\right\rangle = 2i\Delta_c\left\langle \hat{a}^{\dagger 2}\right\rangle + 2i\Omega\left\langle \hat{a}^{\dagger}\right\rangle - \Gamma_a\left\langle \hat{a}^{\dagger 2}\right\rangle + 2ig_{opt}\left[\left\langle \hat{a}^{\dagger 2}\hat{b}^{\dagger 2}\right\rangle + \left\langle \hat{a}^{\dagger 2}\hat{b}^{2}\right\rangle + 2\left\langle \hat{a}^{\dagger 2}\hat{b}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger 2}\right\rangle\right]. \quad (28)$$

$$\frac{d}{dt}\left\langle \hat{b}^{2}\right\rangle = -2i\omega_M\left\langle \hat{b}^{2}\right\rangle - \Gamma_b\left\langle \hat{b}^{2}\right\rangle - 2ig_{opt}\left[\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{2}\right\rangle\right]. \quad (29)$$

$$\frac{d}{dt}\left\langle \hat{b}^{\dagger 2}\right\rangle = 2i\omega_M\left\langle \hat{b}^{\dagger 2}\right\rangle - \Gamma_b\left\langle \hat{b}^{\dagger 2}\right\rangle + 2ig_{opt}\left[\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger 2}\right\rangle\right]. \quad (30)$$

where $\overline{n}_{th}^{a}$, $\overline{n}_{th}^{b}$ are the thermal mean number of cavity photons and membrane oscillations respectively.

2.2 Coupled Equations and Decorrelation of Higher-Order Operators

The above set of equations are not closed, constitute higher order operator products, and need to be solved numerically with approximations. We proceed to decorrelate all the higher-order correlations in the above equations. The above set of equations will then be closed up to second order when we apply the decorrelation method. We proceed to decorrelate the higher-(third-and fourth-) order quantum correlations present in the above equations like [28], which studied the correlation of photon pairs from a double Raman Amplifier; a hybrid quantum optomechanical system containing a single semiconductor quantum well [29]. This approach corresponds to truncation of higher-order operator products in order to solve for all the second-order correlation functions. A similar kind of approximation has also been used in Ref. [30] to study the dynamics of a two-mode BEC beyond mean field approximation:

$$\langle ABC\rangle \approx \left[\langle A\rangle\langle BC\rangle + \langle AB\rangle\langle C\rangle + \langle AC\rangle\langle B\rangle\right] \quad (31)$$

and

$$\langle ABCD\rangle \approx \left[\langle AB\rangle\langle CD\rangle + \langle AC\rangle\langle BD\rangle + \langle AD\rangle\langle BC\rangle\right]. \quad (32)$$

2.3 Temporal Evolutions

After applying the decorrelation method, the above set of equations reduced to a closed set of coupled equations given in Appendix A. These equations contain all possible first-order operator averages and second-order correlation functions. The closed set of equations, given in Appendix A, is further solved numerically for the transient dynamics as well as nonclassical photon statistics.

Figure 1 shows the temporal evolution of the mean number of excitations for the cavity photons $\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle$ and phonons due to mechanical motion of moving membrane $\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle$ for a set of different detuning parameters. For a detuning of the order of mechanical frequency ω_M, cavity is driven around red sideband associated with the membrane oscillator and we have photonic oscillations $\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle$ with phonon sidebands $\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle$. Both mean photonic and phonon excitations are asymmetric in shape also due to strong quadratic optomechanical coupling. As the detuning increases, cavity is driven far from red sideband regime due to which amplitude of the oscillations for $\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle$ are reduced as shown in Figure 1(c). For a very large detuned cavity, oscillations for $\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle$ is completely reduced to its initial value, whereas for $\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle$ we have a regular oscillations over the whole range of time scale. So, for a quadratic coupled optomechanical system cavity should be driven in red (blue) sideband regime of membrane oscillator. This is also required for generation of sub-Poissonian light from the driven cavity in our system Hamiltonian.

In Figure 2, we observe the effect of cavity decay on the temporal dynamics of the system. In the weak cavity decay regime, we have undamped oscillations for photonic as well as phonon oscillations as shown in Figure 2(a). However, in the strong cavity decay regime, amplitude of oscillations for both $\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle$ as well as $\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle$ show damping with time as shown in the Figure 2(b). Although, in both the cases decay rate for phonons Γ_b remains the same. For strong cavity decay, photons leak out of the cavity very rapidly and this leads to damped oscillations for mean phonon number $\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle$ also with time.

We also study the temporal dynamics for finite thermal phonon numbers in Figure 3. We can see that the mean phonon number $\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle$ due to mechanical motion of membrane increases and saturates with time due to thermal noise (for finite $\overline{n}_{th}^{b}$) even for the strong decay regime of cavity as well as mechanical mode as shown in Figure 3(b). So, a finite thermal noise due to phonons is found to increase the mean number of excitations for the dielectric membrane. This is due to dependence of mean excitations number of phonons $\left(\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right)$ on thermal noise and can be also seen in Eq.A6. In this case mean excitations number for photonic and phonons also oscillate in opposite phase upto certain range of time.

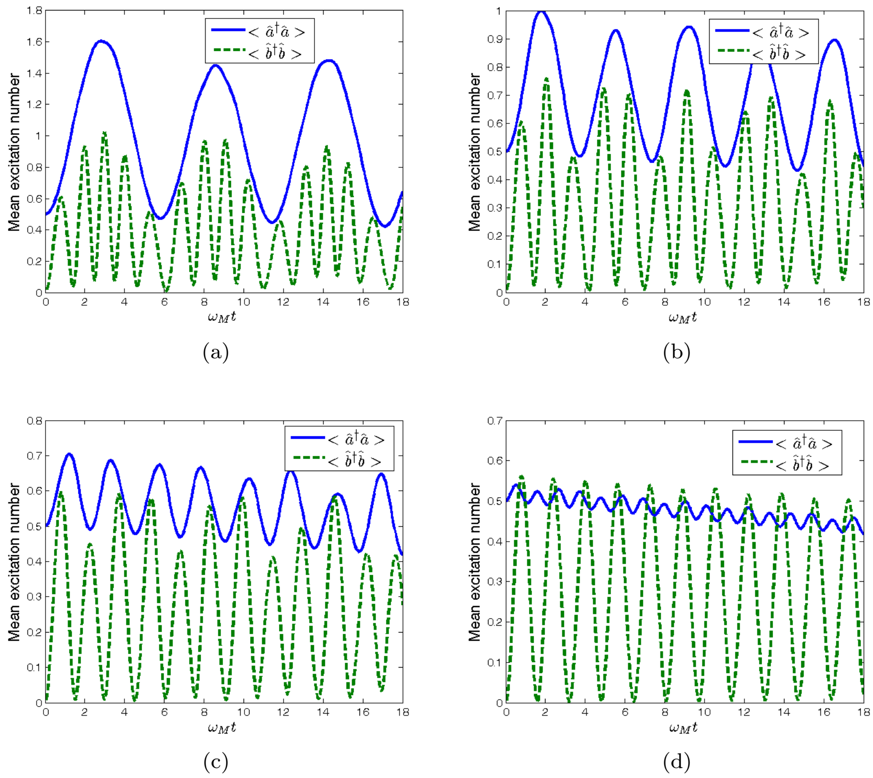
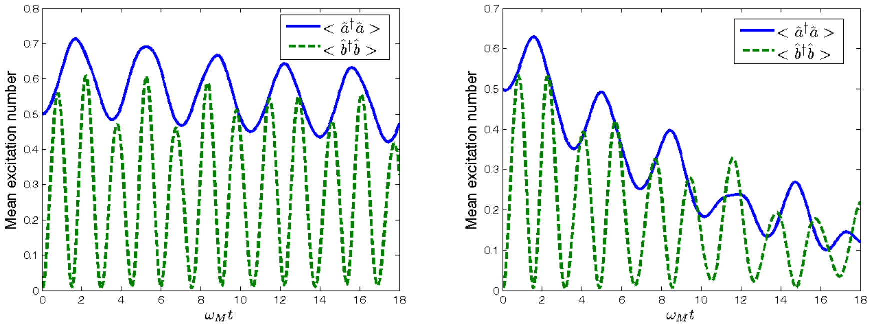
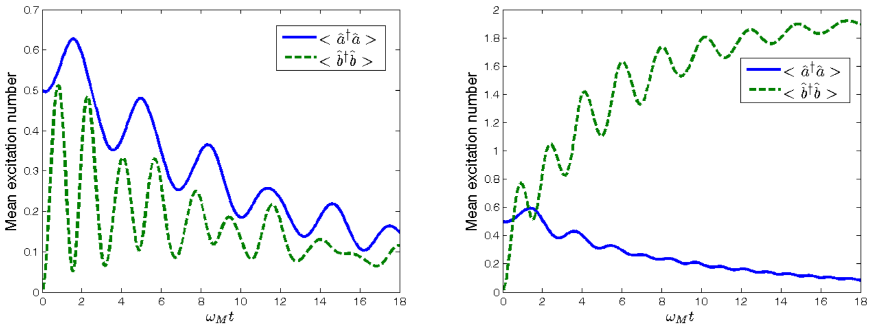


Fig. 1: (Color online) Mean number of cavity photons $\left\langle \hat{a}^\dagger \hat{a} \right\rangle$ (blue line) and phonons of moving membrane $\left\langle \hat{b}^\dagger \hat{b} \right\rangle$ (green line) for chosen set of parameters $g_{opt}/\omega_M = 1.4$; $\mathit{\Gamma}_a/\omega_M = 0.01$; $\mathit{\Gamma}_b/\omega_M = 0.001$; $\mathit{\Omega}/\omega_M = 0.6$; $\overline{n}^a_{th} = \overline{n}^b_{th} = 0$; where ω_M is frequency of mechanical motion of the membrane chosen to 1 in our numerical simulations. a) $\mathit{\Delta}_c/\omega_M = 0.5$; b) $\mathit{\Delta}_c/\omega_M = 1.0$; c) $\mathit{\Delta}_c/\omega_M = 2.0$; d) $\mathit{\Delta}_c/\omega_M = 5.0$.



(a) (b)

Fig. 2: (Color online) Mean number of cavity photons $\left\langle \hat{a}^\dagger \hat{a} \right\rangle$ (blue line) and phonons of moving membrane $\left\langle \hat{b}^\dagger \hat{b} \right\rangle$ (green line) for chosen set of parameters $\Delta_c/\omega_M = 1.0$; $g_{opt}/\omega_M = 1.4$; $\Gamma_b/\omega_M = 0.001$; $\Omega/\omega_M = 0.4$; $\overline{n}^a_{th} = \overline{n}^b_{th} = 0$; a)For weak cavity decay $\Gamma_a/\omega_M = 0.01$; b) For strong cavity decay $\Gamma_a/\omega_M = 0.1$.



(a) (b)

Fig. 3: (Color online) Effects of thermal phonons on the mean number of cavity photons $\left\langle \hat{a}^\dagger \hat{a} \right\rangle$ (blue line) and phonons of moving membrane $\left\langle \hat{b}^\dagger \hat{b} \right\rangle$ (green line) for chosen set of parameters $\Delta_c/\omega_M = 1.0$; $g_{opt}/\omega_M = 1.4$; $\Omega/\omega_M = 0.4$ under strong decay regime $\Gamma_a/\omega_M = \Gamma_b/\omega_M = 0.1$; a) For $\overline{n}^b_{th} = 0.0$; b) $\overline{n}^b_{th} = 2.0$.

3 Photon Statistics

In addition to the mean excitation numbers for cavity $\left(\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\right)$ as well as moving membrane $\left(\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right)$, we have computed the normalized two-boson correlation functions for the cavity field $g_a^{(2)}(0)$, quadratically coupled membrane $g_b^{(2)}(0)$, and the cross-correlation between them $g_{ab}^{(2)}(0)$ using [27,28,29].

$$g_a^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^2\right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle^2} \approx \left[\frac{2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + \left\langle \hat{a}^{\dagger 2}\right\rangle\left\langle \hat{a}^2\right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle^2}\right]. \tag{33}$$

$$g_b^{(2)}(0) = \frac{\left\langle \hat{b}^{\dagger 2}\hat{b}^2\right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle^2} \approx \left[\frac{2\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{b}^{\dagger 2}\right\rangle\left\langle \hat{b}^2\right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle^2}\right]. \tag{34}$$

$$g_{ab}^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{b}\hat{a}\right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle} \approx \left[\frac{\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\hat{a}\right\rangle + \left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}\hat{a}\right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle}\right]. \tag{35}$$

If $g_a^{(2)}(0)$, $g_b^{(2))}(0)$ and $g_{ab}^{(2)}(0)$ satisfy the inequality $g_X^{(2)}(0) < 1$; $X = a, b, ab$, then the statistics of the bosonic systems are referred to as *sub-Poissonian*. Statistics with $g_X^{(2)}(0) = 1$ and $g_X^{(2)}(0) > 1$ are similarly referred to as *Poissonian* and *super-Poissonian,* respectively. Here, we have studied the temporal dynamics of second-order correlation functions, namely the self-correlation and cross correlation for cavity and oscillating membrane modes.

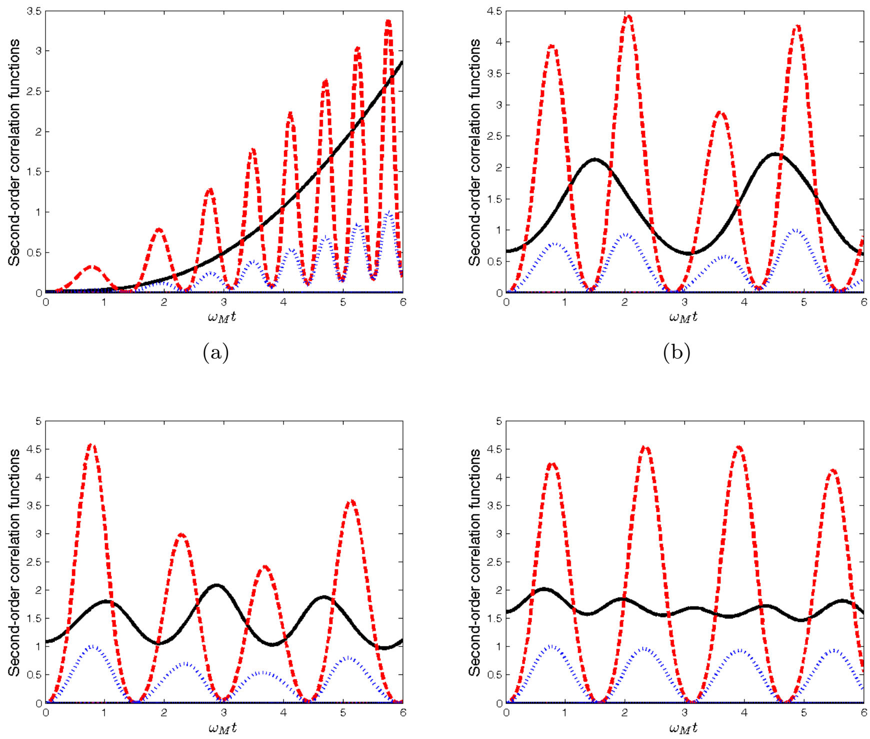
In photon blockade effect, where coupling of a single photon to the system hinders the coupling of the subsequent photons, we have $g_a^{(2)}(0) < 1$ for the cavity mode. Similarly, in photon induced tunneling regime, coupling of initial photons favors the coupling of the subsequent photons and leads to condition $g_a^{(2)}(0) > 1$. We examine the effects of various parameters on the temporal behavior of $g_a^{(2)}(0)$, $g_b^{(2)}(0)$ and $g_{ab}^{(2)}(0)$ as well as under different decay regimes for cavity photons as well as phonons due to micromechanical motion.

The second order correlation functions, namely the self- and cross-correlations for photonic and membrane oscillations are shown in Figure 4. We study the temporal dynamics of $g_a^{(2)}(0)$, $g_b^{(2))}(0)$ and $g_{ab}^{(2)}(0)$ for different cavity detunings Δ_c. It can be seen that for a resonant cavity i.e. $\Delta_c = 0$ and for a shorter period of time $\omega_M t \sim 2$, all the three correlations follow sub-Poissonian photon statistics as shown in Figure 4 (a). As $\omega_M t$ increases, $g_a^{(2)}(0)$ tends to become super-Poissonian, whereas $g_b^{(2)}(0)$ oscillates from sub-Poissonian to super-Poissonian. For a finite detuning $\Delta_c \sim \omega_M$, $g_a^{(2)}(0)$ as well as $g_b^{(2))}(0)$ both oscillate periodically from sub-Poissonian to super-Poissonian whereas $g_{ab}^{(2)}(0)$ mostly remains sub-Poissonian as shown in Figure 4(b). For a very far off detuned cavity, $g_a^{(2)}(0)$ oscillates from Poissonian to super-Poissonian and it never becomes sub-Poissonian, whereas $g_b^{(2))}(0)$ behaves almost the same and $g_{ab}^{(2))}(0)$ oscillates from sub-Poissonian to Poissonian

as shown in Figure 4(c) and Figure 4(d). For a very large cavity detuning, a single photon can not be resonantly excited into the cavity, while the probability of finding two or more photons resonantly enhanced [26]. So, we do not have photon blockade effect for larger cavity detuning at any period of time $\omega_M t$.

Similarly, for a finite detuning Δ_c, we study the effect of optomechanical coupling strengths g_{opt} on various correlations in Figure 5. For the case g_{opt} comparable to driving field Ω, $g_a^{(2))}(0)$ and $g_b^{(2))}(0)$ oscillate from sub-Poissonian to super-Poissonian, whereas $g_{ab}^{(2))}(0)$ remains mostly sub-Poissonian except at some points as shown in Figure 5(a) and Figure 5(b). For a very strong optomechanical coupling g_{opt}, $g_a^{(2))}(0)$ oscillates from Poissonian to super-Poissonian, whereas $g_b^{(2))}(0)$ remains same and $g_{ab}^{(2))}(0)$ varies to Poissonian limit as shown in Figure 5(d).

Furthermore, we examine the effects of photonic and phonon decay rates on various two-boson correlations in Figure 6. For very weak decay rates of cavity and membrane oscillation, $g_a^{(2))}(0)$ varies from Poissonian to super-Poissonian and never becomes sub-Poissonian. Although $g_b^{(2))}(0)$ oscillates from sub-Poissonian to super-Poissonian and $g_{ab}^{(2))}(0)$ remains mostly sub-Poissonian as shown in Figure 6(a). For a strong cavity decay rate Γ_a, $g_a^{(2))}(0)$ becomes sub-Poissonian over a large scale of time and even becomes smaller than 0.5 as shown in Figure 6(b). This means, we have a photon blockade effect only in resolved sideband regime $\Gamma_a/\omega_M < 1$ but not in the deep-resolved-sideband regime $\Gamma_a/\omega_M << 1$. While, for a very strong cavity as well as membrane oscillation decay rates, all the three correlations decay with time rapidly as shown in Figure 6(c). So, strong photonic and phonon decay rates play a positive role for getting sub-Poissonian photon statistics from a driven cavity in our system Hamiltonian.



(a) (b)

(c) (d)

Fig. 4: Second-order autocorrelations $g_a^{(2)}(0)$ (black solid line) for cavity mode, $g_b^{(2)}(0)$ for moving membrane (red dashed line) and $g_{ab}^{(2)}(0)$ (blue dotted line) corresponding cross-correlation between photons and phonons for chosen set of parameters $g_{opt}/\omega_M = 1.5$; $\Gamma_a/\omega_M = 0.01$; $\Gamma_b/\omega_M = 0.001$; $\Omega/\omega_M = 0.6$; $\overline{n}_{th}^a = \overline{n}_{th}^b = 0$; a) $\Delta_c/\omega_M = 0.0$; b) $\Delta_c/\omega_M = 1.3$; c) $\Delta_c/\omega_M = 2.5$; d) $\Delta_c/\omega_M = 4.0$.

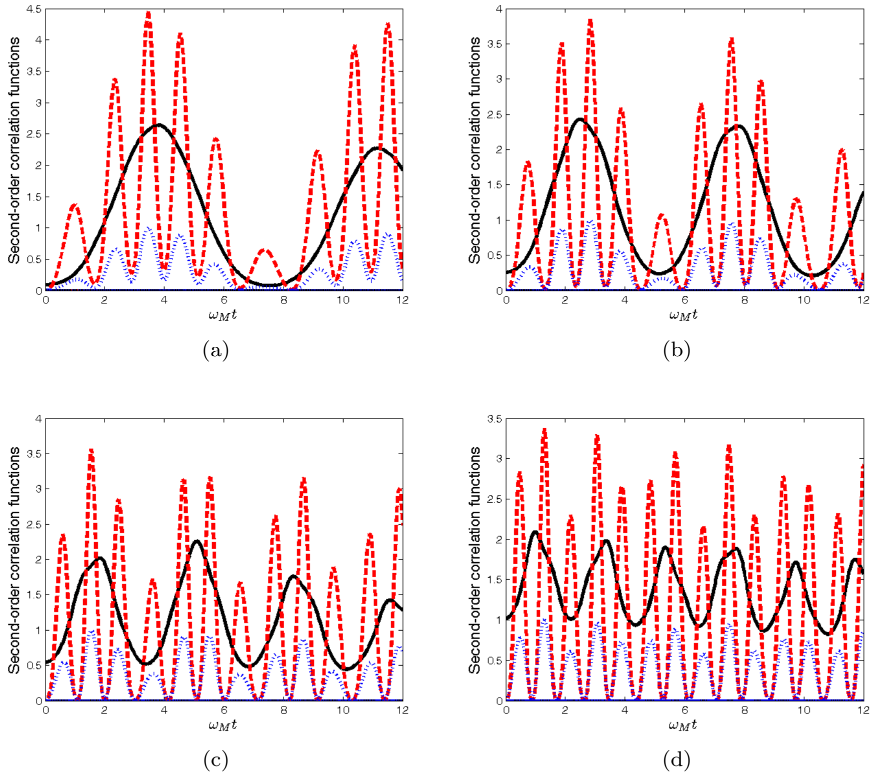


Fig. 5: Second-order autocorrelations $g_a^{(2)}(0)$ (black solid line) for cavity mode, $g_b^{(2)}(0)$ for moving membrane (red dashed line) and $g_{ab}^{(2)}(0)$ (blue dotted line) corresponding cross-correlation between photons and phonons for chosen set of parameters $\Delta_c/\omega_M = 0.5$; $\Gamma_a/\omega_M = 0.01$; $\Gamma_b/\omega_M = 0.001$; $\Omega/\omega_M = 0.6$; $\overline{n}_{th}^a = \overline{n}_{th}^b = 0$; a) $g_{opt}/\omega_M = 0.8$; b) $g_{opt}/\omega_M = 1.7$; c) $g_{opt}/\omega_M = 3.0$; d) $g_{opt}/\omega_M = 5.0$.

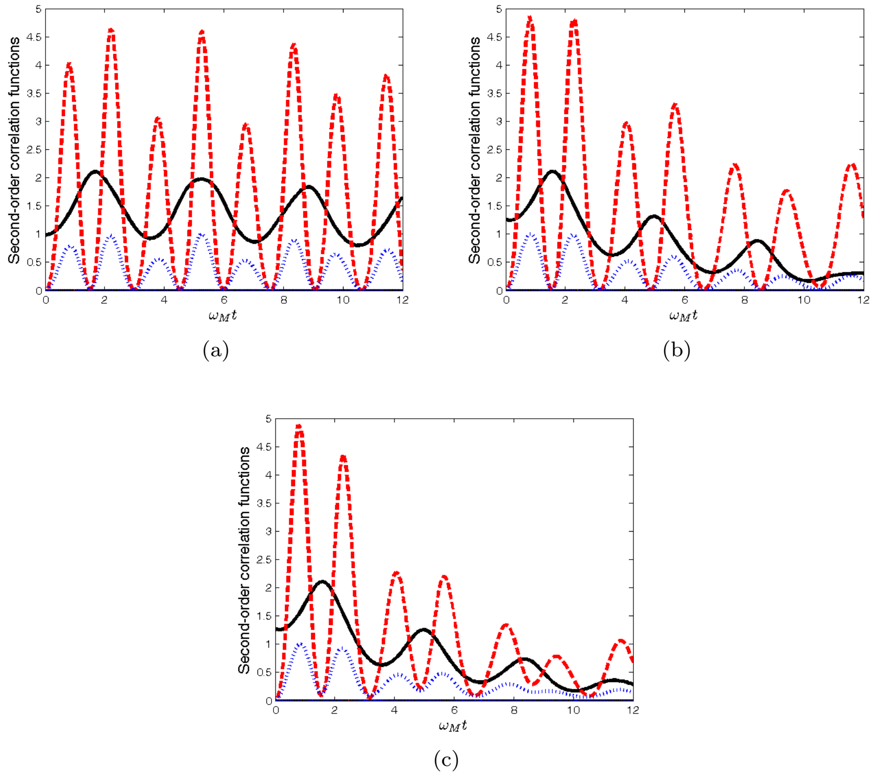


Fig. 6: Second-order autocorrelations $g_a^{(2)}(0)$ (black solid line) for cavity mode, $g_b^{(2)}(0)$ for moving membrane (red dashed line) and $g_{ab}^{(2)}(0)$ (blue dotted line) corresponding cross-correlation between photons and phonons for chosen set of parameters $\Delta_c/\omega_M = 1.0$; $g_{opt}/\omega_M = 1.4$; $\Omega/\omega_M = 0.4$; $\overline{n}_{th}^a = \overline{n}_{th}^b = 0$ under different decay regimes; a) $\Gamma_a/\omega_M = 0.01$, $\Gamma_b/\omega_M = 0.001$; b) $\Gamma_a/\omega_M = 0.1$; $\Gamma_b/\omega_M = 0.001$; c) $\Gamma_a/\omega_M = \Gamma_b/\omega_M = 0.1$.

4 Conclusion

We have studied a qudratically coupled optomechanical system formed by a micropillar with moveable Bragg reflectors and a thin dielectric membrane at the node (or antinode) of the resonator through the Heisenberg- Langevin approach. We have obtained coupled equations up to the second order without any approximation. We decorrelate the higher-order correlation functions in the coupled equations to obtain a closed set of equations. This enables the study of temporal dynamics of the optomechanical system as well as two-boson correlation functions under different scenarios. In the resolved sideband regime, mean number of cavity as well as mechanical excitation show oscillations. Furthermore, amplitude of these oscillations changes as we change the cavity detuning. We have also studied the temporal dynamics under the different decay regimes of cavity and membrane oscillations. We have discussed the effect of thermal noise due to phonons on the dynamics of the optomechanical system. Furthermore, we analyzed the temporal dynamics of the second-order correlation functions, namely the self and cross-correlation of the cavity and mechanical modes under different cavity detunings as well as optomechanical coupling strengths. We obtained the regimes where all the three correlations display strong sub-Poissonian photon statistics particularly at very small cavity detuning as well as strong cavity decay rate. Our study is useful for coherent control of photon statistics as well as photon and phonon correlations in quadratically coupled optomechanical systems.

Acknowledgments

The authors thank University of Malaya for financial support under the University of Malaya Research Grant (RG231-12AFR and RP006A-13AFR). This work also get supported through the postdoctoral fellowship position held by Shailendra Kumar Singh in TÜBİTAK-1001 Grant No. 114F170. SKS would like to gratefully acknowledge Dr. Mehmet Emre Tasgin for his kind help to introduce the subject of cavity optomechanics in details. SKS also kindly acknowledge to Hacettepe university colleague Mr. Saidul Alom Mozumdar for his help to improve manuscript grammatically.

References

1. S. Haroche and J.M. Raimond,"Radiative Properties of Rydberg States in Resonant Cavities", Advances in Atomic and Molecular Physics **20**, 347 (1985).
2. D. Meschede, H. Walther and G. Muller,"One-Atom Maser", Phys. Rev. Lett. **54**, 551 (1985).
3. E.T. Jaynes and F.W. Cummings,"Comparison of quantum and semiclassical radiation theories with application to the beam maser", Proc. IEEE **51,** 89 (1963).
4. E. Yablonovitch,"Inhibited Spontaneous Emission in Solid-State Physics and Electronics", Phys. Rev. Lett. **58**, 2059 (1987).
5. J.M. Gerard, D. Barrier, J.Y. Marzin, R. Kuszelewicz, L. Manin, E. Costard, V. Thierry-Mieg, and T. Rivera, "Quantum boxes a active probes for photonic microstructures: The pillar microcavity case", Appl. Phys. Lett. **69**, 449 (1996).
6. C.C. Lin, M.-C. Wu, B.-W. Shiau, Y.-H. Chen, I.A. Yu, Y.-F. Chen, and Y.-C. Chen, "Enhanced all-optical switching with double slow light pulses", Phys. Rev. A **86**, 063836 (2012).

7. J. Hwang, M. Pototschnig, R. Lettow, G. Zumofen, A. Renn, S. Gotzinger, and V. Sandoghdar, "A single-molecule optical transistor", Nature **460**, 76 (2009).
8. Y.O. Dudin and A. Kuzmich, "Strongly Interacting Rydberg Excitations of a Cold Atomic Gas", Science **336**, 887 (2012).
9. V. Venkataraman, K. Saha, P. Londereo and A.L. Gaeta, "Few-Photon all-Optical Modulation in a Photonic Band-Gap Fiber", Phys. Rev. Lett. **107**, 193902 (2011).
10. P. Kolchin, R.F. Oulton, and X. Zhang, "Nonlinear Quantum Optics ina Waveguide: Distinct Single Photons Strongly Interacting at the Single Atom Level", Phys. Rev. Lett. **106**, 113601 (2011).
11. T.J. Kippenberg and K.J. Vahala, "Cavity Optomechanics: Back- Action at the Mesoscale", Science **321**, 1172 (2008).
12. S. Gröblacher, K. Hammerer, M.R. Vanner, M. Aspelmeyer, "Observation of strong coupling between a micromechanical resonator and an optical cavity field", Nature **460**, 724 (2009).
13. I. Wilson-Rae, N. Nooshi, W. Zwerger, T.J. Kippenberg,"Theory of Ground State Cooling of a Mechanical Oscillator Using Dynamical Backaction", Phys. Rev. Lett. **99**, 093901 (2007).
14. J.D. Thompson, B.M. Zwickl, A.M. Jayich, F. Marquardt, S.M. Girvin, "Strong dispersive coupling of a high-finess cavity to a micromechanical membrane", Nature **452**, 72 (2008).
15. M. Aspelmeyer, S. Gröblacher, K. Hammerer, N. Kiesel, "Quantum optomechanics - throwing a glance", JOSA B, **27**, A189-A197 (2010).
16. E. Verhagen, S. Deleglise, S. Weis, A. Schliesser, and T. J.Kippenberg, "Quantum-coherent coupling of a mechanical oscillator to an optical cavity mode", Nature **482**, 63 (2012).
17. J. Chan, T. P. Mayer Alegre, A. H. Safavi-Naeini, J.T. Hill, A. Krause, S. Gröblacher, M. Aspelmeyer, "Laser cooling of a nanomechanical oscillator into its quantum ground state", Nature **478**, 89 (2011).
18. S. Bose, K. Jacobs and P. L. Knight,"Scheme to probe the decoherence of a macroscopic object", Phys. Rev. A **59**, 3204 (1999).
19. D.H. Santamore, A.C. Doherty and M.C. Cross,"Quantum nondemolition measurement of Fock states of mesoscopic mechanical oscillators", Phys. Rev. B **70**, 144301 (2004).
20. C. Hohberger and K. Karrai,"Cavity cooling of a microlever", Nature **432**, 1002 (2004).
21. O. Arcizet, P.F. Cohadon, T. Briant, M. Pinard, and A. Heidmann, "Radiation- pressure cooling and optomechanical instability of a micromirror", Nature **444**, 71 (2006).
22. I. Martin, and W.H. Zurek,"Measureent of Energy Eigenstates by a Slow Detector", Phys. Rev. Lett. **98**, 120401 (2007).
23. K. Jacobs, P. Lougovski, and M. Blencowe,"Continous Measurement of the Energy Eigenstates of a Nanomechanical Resonator without a Nondemolition Probe", Phys. Rev. Lett. **98**, 147201 (2007).
24. J.D. Thompson, B.M. Zwckl, A.M. Jayich, F. Marquardt, S.M. Girvin, and J.G.E. Harris, "Strong dispersive coupling of a high-finess cavity to a micromechanical membrane", Nature 452, 72 (2008).
25. J.C. Sankay,C. Yang, B.M. Zwickl, A.M. Jayich, and J.G.E. Harris, "Strong and tunable nonlinear optomechanical coupling in a low-loss system ", Nat. Phys. 6, 707 (2010).
26. Jie-Qiao Liao and F. Nori,"Photon blockade in quadratically coupled optomechanical systems ", Phys. Rev. A **88**, 023853 (2013).
27. M.O. Scully and M. Suhail Zubairy, "Quantum Optics", Cambridge University Press 1997.
28. C.H. Raymond Ooi, Q. Sun, M. Suhail Zubairy,and M.O. Scully, "Correlation of photon pairs from the double Raman amplifier: generalized analytical quantum Langevin Theory", Phys. Rev. A **75**, 013820 (2007).
29. S.K. Singh and C.H. Raymond Ooi, "Quantum correlations of quadratic optomechanical oscillator", JOSA B, **31**, 2390-2398 (2014).
30. J.R. Anglin and A. Vardi, "Dynamics of a two-mode Bose-Einstein condensate beyond mean-field approximation", Phys. Rev. A **64**, 013605 (2001).

A Decorrelated and Closed set of Coupled Equations

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= -i\Delta_c\langle\hat{a}\rangle - i\Omega - \frac{1}{2}\Gamma_a\langle\hat{a}\rangle - ig_{opt}\left[\left\{\langle\hat{a}\rangle\left\langle\hat{b}^{\dagger 2}\right\rangle + 2\left\langle\hat{a}\hat{b}^{\dagger}\right\rangle\left\langle\hat{b}^{\dagger}\right\rangle\right\}\right. \\ &\left. + \left\{\langle\hat{a}\rangle\left\langle\hat{b}^{2}\right\rangle + 2\left\langle\hat{a}\hat{b}\right\rangle\left\langle\hat{b}\right\rangle\right\} + 2\left\{\langle\hat{a}\rangle\left\langle\hat{b}^{\dagger}\hat{b}\right\rangle + \left\langle\hat{a}\hat{b}^{\dagger}\right\rangle\left\langle\hat{b}\right\rangle + \left\langle\hat{a}\hat{b}\right\rangle\left\langle\hat{b}^{\dagger}\right\rangle\right\} + \langle\hat{a}\rangle\right].\end{aligned} \tag{36}$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}\right\rangle = i\Delta_c\left\langle \hat{a}^{\dagger}\right\rangle + i\Omega - \frac{1}{2}\Gamma_a\left\langle \hat{a}^{\dagger}\right\rangle + ig_{opt}\left[\left\{\left\langle \hat{a}^{\dagger}\right\rangle\left\langle \hat{b}^{\dagger 2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{\dagger}\right\rangle\right\}\right. \\ \left. + \left\{\left\langle \hat{a}^{\dagger}\right\rangle\left\langle \hat{b}^{2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}\right\rangle\right\} + 2\left\{\left\langle \hat{a}^{\dagger}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\right\rangle\right\} + \left\langle \hat{a}^{\dagger}\right\rangle\right]. \tag{37}$$

$$\frac{d}{dt}\left\langle \hat{b}\right\rangle = -i\omega_M\left\langle \hat{b}\right\rangle - \frac{1}{2}\Gamma_b\left\langle \hat{b}\right\rangle - 2ig_{opt}\left[\left\langle \hat{a}^{\dagger}\right\rangle\left\{\left\langle \hat{a}\hat{b}\right\rangle + \left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\right\}\right. \\ \left. + \left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\{\left\langle \hat{b}\right\rangle + \left\langle \hat{b}^{\dagger}\right\rangle\right\} + \left\langle \hat{a}\right\rangle\left\{\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\right\}\right]. \tag{38}$$

$$\frac{d}{dt}\left\langle \hat{b}^{\dagger}\right\rangle = i\omega_M\left\langle \hat{b}^{\dagger}\right\rangle - \frac{1}{2}\Gamma_b\left\langle \hat{b}^{\dagger}\right\rangle + 2ig_{opt}\left[\left\langle \hat{a}^{\dagger}\right\rangle\left\{\left\langle \hat{a}\hat{b}\right\rangle + \left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\right\}\right. \\ \left. + \left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\{\left\langle \hat{b}\right\rangle + \left\langle \hat{b}^{\dagger}\right\rangle\right\} + \left\langle \hat{a}\right\rangle\left\{\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\right\}\right]. \tag{39}$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle = -i\Omega\left(\left\langle \hat{a}^{\dagger}\right\rangle - \left\langle \hat{a}\right\rangle\right) - \Gamma_a\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + \Gamma_a\bar{n}^{a}_{th}. \tag{40}$$

$$\frac{d}{dt}\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle = -2ig_{opt}\left[\left\{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{b}^{\dagger 2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\right\} - \left\{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{b}^{2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}\hat{b}\right\rangle\right\}\right] \\ -\Gamma_b\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle + \Gamma_b\bar{n}^{b}_{th}. \tag{41}$$

$$\frac{d}{dt}\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle = i\left(\omega_M - \Delta_c\right)\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle - i\Omega\left\langle \hat{b}^{\dagger}\right\rangle - \frac{1}{2}\left(\Gamma_a + \Gamma_b\right)\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle \\ +ig_{opt}\left[2\left\{2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}^{2}\right\rangle\right\} - \left\{\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{2}\right\rangle + 2\left\langle \hat{a}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right\}\right. \\ +2\left\{\left(2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{a}^{2}\right\rangle\right) - \left(2\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle + \left\langle \hat{a}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger 2}\right\rangle\right)\right\} \\ \left. -\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\left(3\left\langle \hat{b}^{\dagger 2}\right\rangle + 1\right)\right]. \tag{42}$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle = i\left(\Delta_c - \omega_M\right)\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle + i\Omega\left\langle \hat{b}\right\rangle - \frac{1}{2}\left(\Gamma_a + \Gamma_b\right)\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle \\ +ig_{opt}\left[\left\{\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger 2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right\} - 2\left\{\left\langle \hat{a}^{\dagger 2}\right\rangle\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\right\}\right. \\ +2\left\{\left(\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right) - \left(\left\langle \hat{a}^{\dagger 2}\right\rangle\left\langle \hat{a}\hat{b}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\right)\right\} \\ \left. +\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left(3\left\langle \hat{b}^{2}\right\rangle + 1\right)\right]. \tag{43}$$

$$\frac{d}{dt}\left\langle \hat{a}\hat{b}\right\rangle = -i\left(\Delta_c + \omega_M\right)\left\langle \hat{a}\hat{b}\right\rangle - i\Omega\left\langle \hat{b}\right\rangle - \frac{1}{2}\left(\Gamma_a + \Gamma_b\right)\left\langle \hat{a}\hat{b}\right\rangle \\ -ig_{opt}\left[2\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle + 3\left\langle \hat{a}\hat{b}\right\rangle\left(\left\langle \hat{b}^{2}\right\rangle + 1\right)\right] - ig_{opt}\left[\left\langle \hat{a}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger 2}\right\rangle + 2\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right] \\ -2ig_{opt}\left[\left(2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{a}^{2}\right\rangle\right) + \left(\left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{2}\right\rangle + 2\left\langle \hat{a}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right)\right. \\ \left. +\left(2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}\hat{b}\right\rangle + \left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{a}^{2}\right\rangle\right)\right]. \tag{44}$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle = i\left(\Delta_c + \omega_M\right)\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle + i\Omega\left\langle \hat{b}^{\dagger}\right\rangle - \frac{1}{2}\left(\Gamma_a + \Gamma_b\right)\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle \\ +ig_{opt}\left[2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle + 3\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left(\left\langle \hat{b}^{\dagger 2}\right\rangle + 1\right)\right] + ig_{opt}\left[\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{2}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right] \\ +2ig_{opt}\left[\left(\left\langle \hat{a}^{\dagger 2}\right\rangle\left\langle \hat{a}\hat{b}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle\right) + \left(\left\langle \hat{b}^{\dagger 2}\right\rangle\left\langle \hat{a}^{\dagger}\hat{b}\right\rangle + 2\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle\right)\right. \\ \left. +\left(2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\right\rangle + \left\langle \hat{a}\hat{b}^{\dagger}\right\rangle\left\langle \hat{a}^{\dagger 2}\right\rangle\right)\right]. \tag{45}$$

$$\frac{d}{dt}\left\langle \hat{a}^2\right\rangle = -2i\Delta_c \left\langle \hat{a}^2\right\rangle - 2i\Omega \left\langle \hat{a}\right\rangle - \Gamma_a \left\langle \hat{a}^2\right\rangle - 2ig_{opt} \left\langle \hat{a}^2\right\rangle \quad (46)$$
$$-2ig_{opt}\left[\left\langle \hat{a}^2\right\rangle \left(\left\langle \hat{b}^{\dagger 2}\right\rangle + \left\langle \hat{b}^2\right\rangle + 2\left\langle \hat{b}^\dagger \hat{b}\right\rangle\right) + 2\left(\left\langle \hat{a}\hat{b}^\dagger\right\rangle^2 + \left\langle \hat{a}\hat{b}\right\rangle^2 + 2\left\langle \hat{a}\hat{b}\right\rangle\left\langle \hat{a}\hat{b}^\dagger\right\rangle\right)\right].$$

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger 2}\right\rangle = 2i\Delta_c \left\langle \hat{a}^{\dagger 2}\right\rangle + 2i\Omega \left\langle \hat{a}^\dagger\right\rangle - \Gamma_a \left\langle \hat{a}^{\dagger 2}\right\rangle + 2ig_{opt} \left\langle \hat{a}^{\dagger 2}\right\rangle \quad (47)$$
$$+2ig_{opt}\left[\left\langle \hat{a}^{\dagger 2}\right\rangle \left(\left\langle \hat{b}^{\dagger 2}\right\rangle + \left\langle \hat{b}^2\right\rangle + 2\left\langle \hat{b}^\dagger \hat{b}\right\rangle\right) + 2\left(\left\langle \hat{a}^\dagger\hat{b}^\dagger\right\rangle^2 + \left\langle \hat{a}^\dagger\hat{b}\right\rangle^2 + 2\left\langle \hat{a}^\dagger\hat{b}^\dagger\right\rangle\left\langle \hat{a}^\dagger\hat{b}\right\rangle\right)\right].$$

$$\frac{d}{dt}\left\langle \hat{b}^2\right\rangle = -2i\omega_M \left\langle \hat{b}^2\right\rangle - \Gamma_b \left\langle \hat{b}^2\right\rangle - 2ig_{opt} \left\langle \hat{a}^\dagger \hat{a}\right\rangle \quad (48)$$
$$-4ig_{opt}\left[\left\langle \hat{a}^\dagger \hat{a}\right\rangle \left(\left\langle \hat{b}^2\right\rangle + \left\langle \hat{b}^\dagger \hat{b}\right\rangle\right) + \left\langle \hat{a}\hat{b}\right\rangle \left(\left\langle \hat{a}^\dagger \hat{b}^\dagger\right\rangle + 2\left\langle \hat{a}^\dagger \hat{b}\right\rangle\right) + \left\langle \hat{a}^\dagger \hat{b}\right\rangle\left\langle \hat{a}\hat{b}^\dagger\right\rangle\right].$$

$$\frac{d}{dt}\left\langle \hat{b}^{\dagger 2}\right\rangle = 2i\omega_M \left\langle \hat{b}^{\dagger 2}\right\rangle - \Gamma_b \left\langle \hat{b}^{\dagger 2}\right\rangle + 2ig_{opt} \left\langle \hat{a}^\dagger \hat{a}\right\rangle \quad (49)$$
$$+4ig_{opt}\left[\left\langle \hat{a}^\dagger \hat{a}\right\rangle \left(\left\langle \hat{b}^{\dagger 2}\right\rangle + \left\langle \hat{b}^\dagger \hat{b}\right\rangle\right) + \left\langle \hat{a}^\dagger\hat{b}^\dagger\right\rangle \left(\left\langle \hat{a}\hat{b}\right\rangle + 2\left\langle \hat{a}\hat{b}^\dagger\right\rangle\right) + \left\langle \hat{a}^\dagger \hat{b}\right\rangle\left\langle \hat{a}\hat{b}^\dagger\right\rangle\right].$$